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Abstract

The generalized Fourier Transform on a given group is applied to invariant feature extraction in the case of a gray-level image. Thus, a new complete and convergent set of invariant features under planar similarities is proposed using the Analytical Fourier–Mellin Transform (AFMT). So, this set gives a distance between the shapes which is invariant under similarities.

Keywords: Completeness; Invariant features; Similarity; Generalized Fourier transform; Analytical Fourier–Mellin transform

1. Introduction

In some cases, a scene description needs to be invariant under certain geometrical transformations of the Euclidean space. The invariant description is usually achieved with the moment invariants, Fourier descriptors, Fourier–Mellin descriptors which have been the subject of numerous papers. A useful general-purpose pattern invariant description method in computation vision should make accurate and reliable recognition of an object possible. Therefore, such a description should necessarily satisfy a number of criteria. The following is a non-exhaustive list of such criteria:

1. A fast *computation* (or to minimize the computation time needed).
2. A good numerical *approximation.*
3. A powerful *discrimination.*
4. A *completeness* of the description which guarantees that if two images have the same invariant representation then the shapes they represent should also be similar.
5. A *distance* between shapes which is invariant under a class of geometric transformations.
6. A *stability* criterion which guarantees that if two invariant representations have a small difference, the objects they represent should also have a small shape difference.

In the literature, authors usually discuss the first three criteria which are very important.

The completeness criterion was first introduced by Crimmins (1982) and mentioned by other authors (Gauthier et al., 1991; Arbter et al., 1990; Ghorbel, 1992a). Crimmins (1982) gave a counterexample for the Fourier descriptors based on the modulus and he proposed a new complete set of invariant parameters. The mathematical definition of the stability criterion was specified in the case of closed contours by Ghorbel (1992b) where a complete and stable set of invariants was proposed. This property implies the existence of a natural Euclidean distance between shapes.

In this paper, we intend to derive a complete set of shape descriptors for gray-level images, which are in-

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variant under similarity transformations. At first, we recall the useful results of the commutative harmonic analysis in order to introduce an invariance approach based on the generalized Fourier Transform on a given abelian group. Such a general approach presents two advantages:

– It elucidates the completeness criterion for an invariant description.

– It can solve more generalized questions: invariance under other classes of transformations (for example: the affine transformations, the projective transformations, ...). In this sense, Segman et al. (1992) introduce an approach based on the theory of Lie groups where the transformation group could be nonlinear.

At last, due to the divergence of the Fourier–Mellin Transform (FMT) near the origin, we propose a new approach by defining the Analytical Fourier–Mellin Transform (AFMT) and thereby a new complete invariant description will be proposed. We underline the implementation difficulties (polar coordinates, the approximation of the AFMT, ...).

The proposed set of invariant parameters is convergent, so it gives a distance between the shapes which is invariant under similarity transformations. However, the complexity of the mathematical nature of this set of invariants does not help us in the theoretical verification of the stability criterion.

2. Some results of the commutative harmonic analysis

Throughout this paper, we shall denote the additive group of real numbers as $\mathbb{R}$, the multiplicative group of nonzero [respectively positive] real numbers as $\mathbb{R}^*$ [respectively $\mathbb{R}^*_+$], the unit circle of the complex plane $\mathbb{C}$ as $\mathbb{S}^1$, the group of positive planar similarities as SimO^+ and the multiplicative group of nonzero complex numbers as $\mathbb{C}^*$. With their topology these groups have a topological group structure. Finally, we remark that all these groups are locally compact and abelian. For this, we limit our studies in this paper to the commutative harmonic analysis. We denote by $L^1(G,\mu)$ the normed vector space of complex functions defined on G.

$$f \text{ is in } L^1(G,\mu) \Leftrightarrow \int_G |f(x)|\,\mathrm{d}\mu(x) < +\infty$$

2.1. Haar measure

Definition 1. Let G be a locally compact abelian group. Then μ is an *invariant measure* on G if and only if $\mu(aB)=\mu(B)$, for every a in G and B a Borel set of G.

This implies that for every function f in $L^1(G,\mu)$

$$\int_G f(ax)\,\mathrm{d}\mu(x) = \int_G f(x)\,\mathrm{d}\mu(x)\ .$$

Definition 2. μ is a *Haar measure* on G if and only if it is positive and invariant.

Haar Theorem (1932). *In every locally compact abelian group G, there exists a unique normalized Haar measure.*

Example. (1) The normalized Haar measure in the real vector space $\mathbb{R}^n$ is the Lebesgue one,

$$\mathrm{d}\mu(x_1, x_2, ..., x_n) = \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n\ .$$

(2) The Haar measure of the multiplicative group $\mathbb{R}^*$ is

$$\mathrm{d}\mu(x) = \mathrm{d}x/x\ .$$

2.2. A group representation

Definition 3. Let H be a vector space. T is a *representation* of G if:

1. $T(x)$ is an endomorphism of H.
2. $T(e) = \mathrm{Id}_H$ where Id_H is the identical operator of H and e is the identity element of G.
3. $T(xy) = T(x)T(y)$ for all x and y in G.
4. T is continuous on G.

Definition 4. T is *irreducible* if and only if H has not a proper subspace S invariant with respect to T $(T(S) \not\subset S)$.

Definition 5. T is *unitary* if and only if the matrix $T(x)$ is unitary for all x in G.

Definition 6. The set of all irreducible and unitary representations of G that we denote $\hat{G}$ is called the *dual* of G.

Proposition 1. *If G is abelian then $\hat{G}$ the dual of G is a locally compact and abelian group and any irreducible and unitary representation becomes a scalar. Then there exists a unique normalized Haar measure $\mu_{\hat{G}}$ in $\hat{G}$.*

Examples. (1) If $G=\mathbb{R}$ then $\hat{G}=\mathbb{R}$ and $T_\lambda(x)=\mathrm{e}^{\mathrm{i}\lambda x}$.
(2) If $G=\mathbb{S}^1$ then $\hat{G}=\mathbb{Z}$ and $T_n(x)=\mathrm{e}^{\mathrm{i}nx}$.
(3) If $G=\mathbb{R}^*$ or $\mathbb{R}^*_+$ then $\hat{G}=\mathbb{R}$ and $T_\alpha(x)=x^{\mathrm{i}\alpha}$.

2.3. Fourier transform on a group

Definition 7. Let f be in $L^1(G,\mu)$ where G is assumed abelian. The *Fourier Transform* on G is defined by:

$$\hat{f}(\lambda)=\int_G f(x)\,[T_\lambda(x)]^{-1}\,\mathrm{d}\mu(x) \tag{1}$$

where μ is the normalized Haar measure of G and $T_\lambda(x)$ is an irreducible and unitary representation of G.

Examples. (1) If $G=\mathbb{R}^n$ then $\hat{G}=\mathbb{R}^n$, and for all λ in $\mathbb{R}^n$

$$\hat{f}(\lambda)=\int_{\mathbb{R}^n} f(x)\,\mathrm{e}^{-\mathrm{i}\langle x,\lambda\rangle}\,\mathrm{d}x$$

is a multi-dimensional Fourier Transform.
(2) If $G=\mathbb{S}^1$ then $\hat{G}=\mathbb{Z}$, and for all n in $\mathbb{Z}$

$$\hat{f}(n)=\int_{[0,2\pi]} f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta$$

is the Fourier coefficient for a 2π periodic function.
(3) If $G=\mathbb{Z}/N\mathbb{Z}$ (the cyclic group of integers) then $\hat{G}=\mathbb{Z}/n\mathbb{Z}$, and for all n in $\mathbb{Z}/N\mathbb{Z}$

$$\hat{f}(n)=\frac{1}{N}\sum_{p=0}^{p=N-1} f(p)\,\mathrm{e}^{2\mathrm{i}\pi np/N}$$

is the Discrete Fourier Transform (DFT).
(4) If $G=\mathbb{R}^*_+$ then $\hat{G}=\mathbb{R}$, and for all λ in $\mathbb{R}$

$$\hat{f}(\lambda)=\int_{\mathbb{R}^*_+} f(x)\,x^{-\mathrm{i}\lambda}\frac{\mathrm{d}x}{x}=\int_0^{+\infty} f(x)\,x^{-\mathrm{i}\lambda-1}\,\mathrm{d}x$$

is the Mellin Transform.

2.4. Inverse Fourier transform

Proposition 2. *If $\hat{f}$ belongs to $L^1(\hat{G},\mu_{\hat{G}})$, then the Inverse Fourier Transform exists and it is defined by:*

$$f(x)=\hat{\hat{f}}(x)=\int_{\hat{G}} \hat{f}(\lambda)\,[T_\lambda(x)]\,\mathrm{d}\mu_{\hat{G}}(\lambda) \tag{2}$$

where $\mu_{\hat{G}}$ is the normalized Haar measure of $\hat{G}$ and $T_\lambda(x)$ is an irreducible and unitary representation of G.

Proof. When G is abelian, $\hat{G}$ becomes a locally compact and abelian group, then there exists a normalized Haar measure $\mu_{\hat{G}}$ in $\hat{G}$ (Haar Theorem) and then any irreducible and unitary representation of the dual group becomes a scalar (Proposition 1). Then the Fourier Transform on the dual group $\hat{G}$ exists and can be defined by:

$$\hat{g}(x)=\int_{\hat{G}} g(\lambda)\,[T_\lambda(x)]^{-1}\,\mathrm{d}\mu_{\hat{G}}(\lambda)\ .$$

Then the Inverse Fourier Transform can be obtained by the formula given in Proposition 2. For more details see (Dieudonné, 1974). □

Example. If $G=\mathbb{R}^*_+$ then $\hat{G}=\mathbb{R}$, and for all x in $\mathbb{R}^*_+$

$$f(x)=\frac{1}{2\mathrm{i}\pi}\int_{-\infty}^{+\infty}\hat{f}(\lambda)\,x^{\mathrm{i}\lambda}\,\mathrm{d}\lambda$$

is the inverse Mellin transform.

3. A complete invariant description

3.1. Analytical Fourier–Mellin transform

It is well known that the direct similarity group on the plane is equivalent to the space of polar coordinates:

$$\Pi=\{(r,\theta)\mid r>0 \text{ and } 0<\theta<2\pi\}\ .$$

The following multiplication is defined in Π

$$(r,\theta)\cdot(r',\theta')=(r\cdot r',\theta+\theta')\ .$$

Π is a locally compact and abelian group. The normalized Haar measure is

$$\mathrm{d}\mu(r,\theta)=\mathrm{d}r/r\,\mathrm{d}\theta$$

and the dual group of Π is $\mathbb{R}\times\mathbb{Z}$. So, the Fourier Transform on Π will be defined as:

$$\hat{f}(k,v)=M_f(k,v)=\int_0^{+\infty}\int_0^{2\pi} f(r,\theta)\,\mathrm{e}^{-\mathrm{i}k\theta} r^{-\mathrm{i}v}\frac{\mathrm{d}r}{r}\,\mathrm{d}\theta\,,$$

for $k\in\mathbb{Z}$ and $v\in\mathbb{R}$. (3)

It is the Fourier–Mellin Transform of the irradiance distribution $f(r, \theta)$ in a two-dimensional image expressed in polar coordinates. The origin of the polar coordinates can be taken in the image center of gravity in order to obtain invariance under translations.

The integral (3) diverges in general, since the convergence is indeed under the assumption that $f(r, \theta)$ is equivalent to Kr^{α} ($\alpha>0$ and K a constant) in a neighbourhood of the origin (the center of gravity of the observed image). For this reason, we consider the Analytical Fourier Mellin Transform that we define here by

$$M_f(k,s=\sigma_0+\mathrm{i}v)=\int_0^{+\infty}\int_0^{2\pi} f(r,\theta)\,\mathrm{e}^{-\mathrm{i}k\theta} r^{\sigma_0+\mathrm{i}v}\frac{\mathrm{d}r}{r}\,\mathrm{d}\theta\,,$$

for $\sigma_0>0$, $k\in\mathbb{Z}$ and $v\in\mathbb{R}$. (4)

This integral converges for the positive values of the real part of the complex number s.

From the commutative harmonic analysis, we can obtain the Inverse of the FMT (IFMT):

$$f(r,\theta)=\sum_k\int_{\mathbb{R}} M_f(k,\sigma_0+\mathrm{i}v)\,\mathrm{e}^{\mathrm{i}k\theta} r^{\sigma_0+\mathrm{i}v}\,\mathrm{d}v\,. \quad (5)$$

This is important for the definition of a complete set of invariant features and for the reconstruction.

The relation between the Analytical Fourier Mellin Transform (AFMT) of two images f and g having the same shape ($g(r,\theta)=f(\alpha r,\theta+\beta)$), where α is a scale factor and β is a rotation parameter, is given by the following equations

$$M_g(k,s=\sigma_0+\mathrm{i}v)=\alpha^{-s}\,\mathrm{e}^{\mathrm{i}k\beta}\,M_f(k,s) \quad (6)$$

for every integer k, for all real numbers v and for a fixed positive value σ_0.

3.2. A complete invariant description

We use the AFMT without losing image information. Therefore, we propose the following solution which will be formulated in the following theorem.

Theorem 1. *Let k be an integer and let s be a complex number such that its real part is strictly positive and fixed. Suppose that $M_f(1,1)$ and $M_f(0,1)$ are not zero. Then the following sequence of complex-valued functions:*

$$I_f(0,1)=|M_f(1,1)|\,[M_f(0,1)]^{-1}\,,$$
$$I_f(k,s=\sigma_0+\mathrm{i}v)$$
$$=[M_f(k,s)]\,[M_f(0,1)]^{-s+k}[M_f(1,1)]^{-k}$$

is a complete set of invariant features under positive and planar similarities.

Proof.
1. *Invariance*

$$M_f(k,\sigma_0+\mathrm{i}v)=\alpha^{-\sigma_0-\mathrm{i}v}\,\mathrm{e}^{\mathrm{i}k\beta}\,M_g(k,\sigma_0+\mathrm{i}v)$$

for $\sigma_0>0$, k integer and v real number.

$$I_f(0,1)=\frac{|M_f(1,1)|}{M_f(0,1)}=\frac{|\mathrm{e}^{\mathrm{i}\beta}\alpha^{-1}M_f(1,1)|}{\alpha^{-1}M_f(0,1)}=I_g(0,1)\,,$$

$$I_f(k,s=\sigma_0+\mathrm{i}v)=[\alpha^{-s}\mathrm{e}^{\mathrm{i}k\beta}M_g(k,s)]$$
$$\times[\alpha^{-1}M_g(0,1)]^{-s+k}[\alpha^{-1}\mathrm{e}^{\mathrm{i}\beta}M_g(1,1)]^{-k}$$
$$=[M_g(k,s)]\,[M_g(0,1)]^{-s+k}[M_g(1,1)]^{k}=I_g(k,s)\,.$$

2. *Completeness*

$$M_f(k,s)=I_f(k,s)\,[M_f(1,1)]^{k}[M_f(0,1)]^{s-k}$$
$$=I_f(k,s)\,[I_f(0,1)]^{k}\mathrm{e}^{\mathrm{i}k\beta}\alpha^{s}$$

where $\alpha=M_f(0,1)$ and $\beta=\arg M_f(1,1)$. The set of invariant features is complete since it is expressed only according to the AFMT's, a given scale factor α and a rotation parameter β. □

A central problem in image analysis and in computer vision is determining the extent to which one shape differs from another. The correlation and template matching can be viewed as techniques for determining the difference between shapes. In order to meet this goal, we propose to use the invariant descriptions. We define a distance between images which is invariant with respect to certain transformations and we precise in the following section the existence conditions of an invariant distance.

4. An invariant metric

Definition 8. We say that two images O_1 and O_2 *have the same shape* if and only if $s(O_1)=O_2$, where s is a positive planar similarity transformation.

The relation "*have the same shape*" defines in the space of all images $\boldsymbol{O}$ an equivalence relation, since the set of planar similarities is a group.

Definition 9. A *shape* is an equivalence class of objects in relation. So, the shape space S is the quotient space of $\boldsymbol{O}$ by the group of positive similarities $\mathrm{SimO}^+(\mathbb{R}^2)$.

Definition 10. A set of scalars $\{I_f(k,s)\}$ is *convergent* if there exists a real number $p>1$ such that $\{I_f(k,s)\}$ belongs to $L^p(\mathbb{Z}\times\mathbb{R},\mu)$, that is to say

$$N_p(f)=\left(\int_{-\infty}^{+\infty}\sum_{k\in\mathbb{Z}}\left|I_f(k,s=\sigma_0+\mathrm{i}v)\right|^p \mathrm{d}v\right)^{1/p}<+\infty\,.$$

Proposition 3. *The existence of a complete and convergent set of invariant features $\{I(k,s)\}$ implies that the shape space S is a metric space with one of the following set of metrics*

$$d_p(G,H)=$$
$$\left(\int_{-\infty}^{+\infty}\sum_{k\in\mathbb{Z}}\left|I_g(k,\sigma_0+\mathrm{i}v)-I_h(k,\sigma_0+\mathrm{i}v)\right|^p \mathrm{d}v\right)^{1/p}$$

where $p>1$, g and h are two images having the shape G and H, respectively.

Proof. d_p is a metric if and only if the following properties are satisfied for all G, H and K in S:

1. $0\leqslant d_p(G,H)<+\infty$,
2. $H=G \Rightarrow d_p(G,H)=0$,
3. $d_p(G,H)=0 \Rightarrow H=G$,
4. $d_p(G,H)\leqslant d_p(G,K)+d_p(K,H)$,
5. $d_p(G,H)=d_p(H,G)$.

The first property is verified because of the convergence of the set of invariant features. The second one comes from the invariance property. The third is satisfied because the set is complete. Finally, the fourth and fifth properties come from the definition of the metric d_p. □

Unfortunately, the set of invariant features proposed in Theorem 1 diverges (does not converge in the sense of Definition 10). So it can not give a metric in the shape space.

Example. Let f be a binary image which represents a half of a disc-shape and let R be its radius,

$$f(r,\theta)=1 \quad \text{for } (r,\theta)\in[0,R]\times[0,\pi],$$
$$=0 \quad \text{otherwise}\,.$$

The AFMT of this image is expressed by

$$M_f(k,s)=\int_0^R\int_0^\pi \mathrm{e}^{-\mathrm{i}k\theta}r^{s-1}\,\mathrm{d}r\,\mathrm{d}\theta\,.$$

For $k\neq 0$, $\quad M_f(k,s)=\dfrac{\mathrm{i}\,R^s[(-1)^k-1]}{sk}$,

otherwise $\quad M_f(0,s)=\pi R^s/s$.

Then the set of invariant features defined in Theorem 1 becomes for $k\neq 0$

$$I_f(k,s)=\frac{\mathrm{i}\,R^s[(-1)^k-1](\pi R)^{k-s}}{sk(-2\mathrm{i}R)^k}=\frac{\mathrm{i}^{k+1}\pi^{k-s}[(-1)^k-1]}{2^k sk}$$

and

$$I_f(0,1)=2/\pi\,.$$

So, the modulus of $I_f(k,s)$ becomes

$$|I_f(2k+1,s)|=\frac{\pi^{2k+1-\sigma_0}}{|s|\,2^{2k+1}(2k+1)} \quad\text{and}$$

$$|I_f(2k,s)|=0\,.$$

Then this set diverges in the sense of Definition 10.

$$(N_p(I_f))^p=\int_{-\infty}^{\infty}\sum_{k\in\mathbb{Z}}|I_f(2k+1,\sigma_0+\mathrm{i}v)|^p\,\mathrm{d}v$$
$$=\int_{-\infty}^{\infty}\sum_{k\in\mathbb{Z}}\left(\frac{\pi^{2k+1-\sigma_0}}{|\sigma_0+\mathrm{i}v|\,2^{2k+1}(2k+1)}\right)^p\mathrm{d}v\,.$$

Using Fubini's theorem for measurable positive-valued functions

$$= \frac{\pi^{(1-\sigma_0)p}}{2^p} \times \int_{-\infty}^{\infty} (\sigma_0^2+v^2)^{-p/2}\,\mathrm{d}v \sum_{k\in\mathbb{Z}} \left(\frac{\pi}{2}\right)^{2kp} \frac{1}{(2k+1)^p}.$$

We denote by $C(p)$ the following integral

$$C(p)=\frac{\pi^{(1-\sigma_0)p}}{2^p}\int_{-\infty}^{\infty}(\sigma_0^2+v^2)^{-p/2}\,\mathrm{d}v$$

which is at least strictly positive, so

$$(N_p(I_f))^p = C(p)\sum_{k\in\mathbb{Z}}\left(\left(\frac{\pi}{2}\right)^{2k}\frac{1}{(2k+1)}\right)^p=+\infty .$$

Then $N_p(I_f)$ diverges, since

$$\left[\left(\frac{\pi}{2}\right)^{2k}\frac{1}{(2k+1)}\right]^p \to +\infty \quad \text{as } k\to\infty .$$

In order to define a distance on the shape space S, we propose in the following theorem, the definition of a new complete and convergent set of invariant primitives.

Theorem 2. *Under the assumptions of Theorem* 1 *and assuming that* $\{M_f(k,s)\}$ *is a convergent set in the sense of Definition* 10, *then*

$$I_f(k,s)$$
$$=M_f(k,s)\,[M_f(1,1)]^{-k}|M_f(1,1)|^k[M_f(0,1)]^{-s}$$

is a complete and convergent set of invariant features under positive and planar similarities.

Proof of convergence. The p-norm of the sequence of invariants can be expressed as:

$$N_p(I_f)=\left(\sum_{k\in\mathbb{Z}}\int_{\sigma_0-\mathrm{i}\infty}^{\sigma_0+\mathrm{i}\infty}|I_f(k,\sigma_0+\mathrm{i}v)|^p\,\mathrm{d}v\right)^{1/p}$$

$$=[M_f(0,1)]^{-\sigma_0}\left(\sum_{k\in\mathbb{Z}}\int_{\sigma_0-\mathrm{i}\infty}^{\sigma_0+\mathrm{i}\infty}|M_f(k,\sigma_0+\mathrm{i}v)|^p\,\mathrm{d}v\right)^{1/p}$$

since:

$$|I_f(k,\sigma_0+\mathrm{i}v)|=|M_f(k,s)|\,[M_f(0,1)]^{-\sigma_0}.$$

This implies that

$$N_p(I_f)=[M_f(0,1)]^{-\sigma_0}N_p(M_f)\ .$$

The set of invariant features is convergent since σ_0 is a fixed number. □

In order to illustrate the result of Theorem 2, let us consider the example of an image which represents the half of a disc-shape. The set of invariant features defined in this theorem becomes:

$$\text{for } k\neq 0,\quad I_f(k,s)=\frac{\mathrm{i}^{k+1}\pi^{-s}[(-1)^k-1]}{sk},$$

$$\text{otherwise}\quad I_f(0,1)=2/\pi .$$

Then the expression of its modulus can be written as

$$|I_f(2k+1,s)|=\frac{2\pi^{-\sigma_0}}{|s|\,(2k+1)} \quad\text{and}$$

$$|I_f(2k,s)|=0 .$$

Thus, this set is convergent in the sense of $L^2(\mathbb{Z}\times\mathbb{R},\mu)$, since:

$$N_2(I_f)=2\pi^{-\sigma_0}\left(\sum_{k\neq 0}\frac{1}{(2k+1)^2}\right)^{1/2}\left(\int_{-\infty}^{+\infty}\frac{1}{\sigma_0^2+v^2}\,\mathrm{d}v\right)^{1/2}$$
$$=\frac{2\pi^{-\sigma_0+1/2}}{\sqrt{\sigma_0}}\left(\sum_{k\neq 0}\frac{1}{(2k+1)^2}\right)^{1/2}<+\infty .$$

This proves the convergence in the sense of $L^2(\mathbb{Z}\times\mathbb{R},\mu)$.

5. Numerical considerations

The AFMT is expressed in polar coordinates. Its computation must be done with interpolation since numerical images are often presented in Cartesian coordinates. The discrete polar coordinates are formed by centred discrete circles with the same origin. For the computation, we use the integer radius: $r=1,2,3,\ldots,R_{\max}$.

Unfortunately, these circles do not have the same number of realistic points. For any circle, we sample it uniformly with the same step, independently from the circle. We propose the numerical approximation of the AFMT by the following formula:

$$\hat{M}_f(k,\sigma_0+\mathrm{i}v)=$$

$$\sum_{r=1}^{R_{\max}} \sum_{m=0}^{N(r)-1} f\left(r, \frac{2\pi m}{N(r)}\right) r^{\sigma_0+\mathrm{i}v-1} \exp -\left\{\frac{2\mathrm{i}\pi mk}{N(r)}\right\}$$

for all k in $\{0, ..., N(r)-1\}$ and m in $\{1, ..., R_{\max}\}$.

This formula does not represent a discrete transform but only an approximation of the AFMT.The IAFMT can be approximated by:

$$f\left(r, \frac{2\pi m}{N(r)}\right) = \sum_{v=-\infty}^{v=+\infty} \sum_{k=0}^{N(r)-1} \hat{M}_f(k, \sigma_0+\mathrm{i}v)\, r^{\sigma_0+\mathrm{i}v} \exp\left\{\frac{2\mathrm{i}\pi mk}{N(r)}\right\}.$$

6. Experimental results and discussion

In this experimental part, we deal with a segmented image of the retina. By using the approximation of the AFMT, we deduce a complete and convergent set of invariant features. This invariant description allows us to define a distance between two retina images of the same person and discriminate others with these primitives. It allows the detection of some anomalous evolution of the retina. When we consider a dynamic sequence of retina images, the movement of vessels can be modelised by approximating the composition of a shift, a rotation, a dilatation and a small nonlinear distortion. Thus, the complete invariant description allows the reconstruction with the elimination of the movement. With the stability property, the set of invariants becomes robust under the small nonlinear distortions between two successive images.

Fig. 1 gives the original image, Fig. 2 represents the original contour retina image in polar coordinates, Fig. 3 gives the modulus of the complete and convergent set of invariant parameters of the original contour image expressed in polar coordinates (v is assimilated as a radius and k as an angle) and Fig. 4 shows the reconstruction of the original contour image by the same set of invariant features using the completeness and the inversion of the AFMT. Experimentally this figure illustrates that the invariant description proposed in this paper is complete since we are able to reconstruct the original contour retina image.

In conclusion, the theory of the commutative harmonic analysis was used in order to define a complete invariant description in the case of gray-level

Fig. 1.

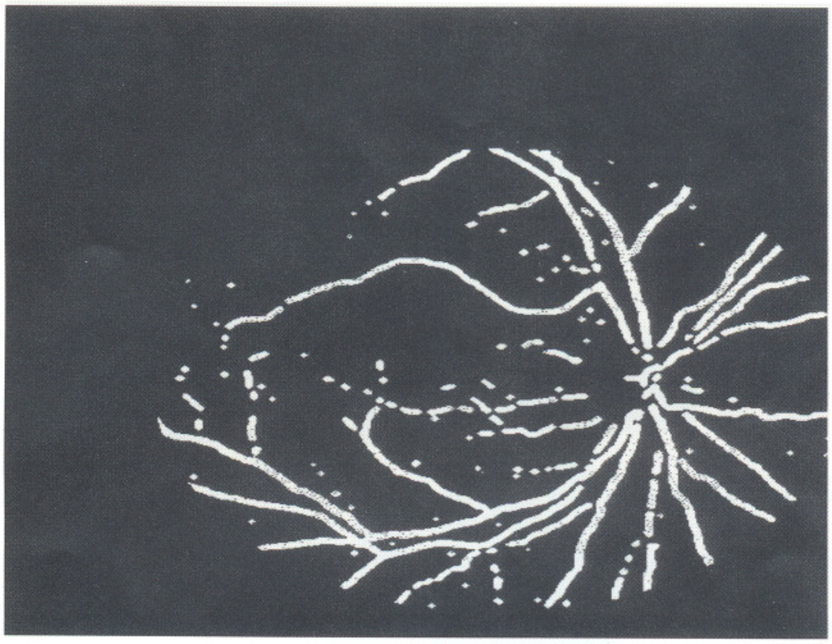

Fig. 2.

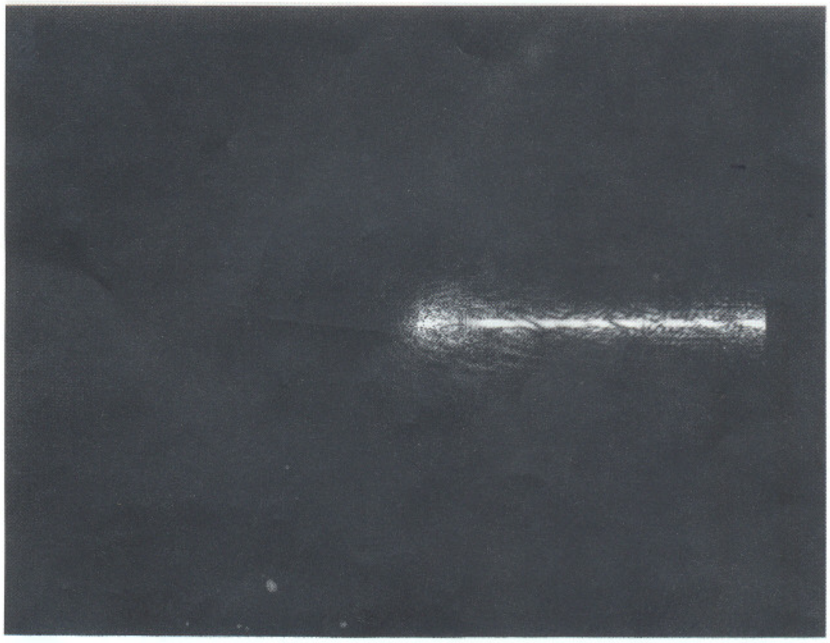

Fig. 3.

images. The set of invariant features is complete and convergent. It allows the definition of a distance between the gray-level shapes. This distance emphasises the independence of differences of rotation, scale factors and camera positions between two images. In the medical context, the invariant distance between

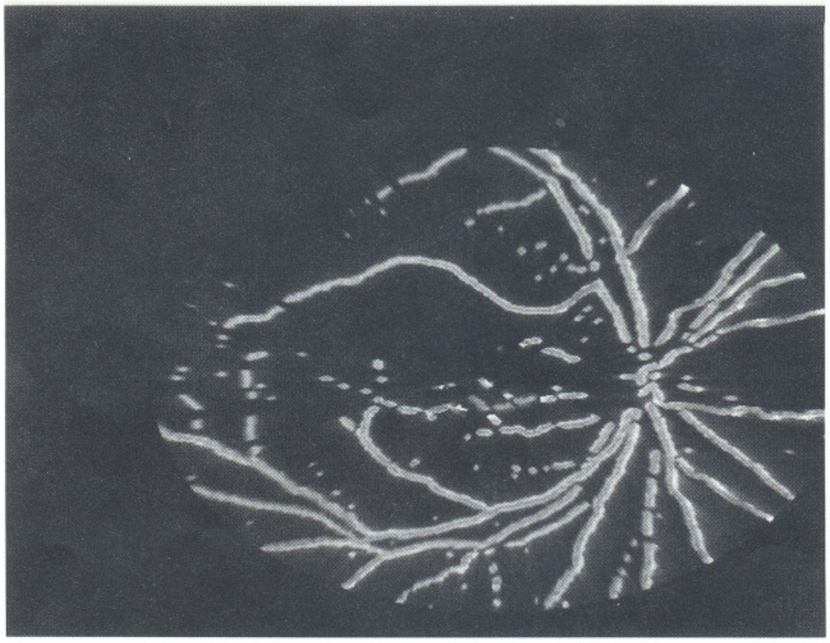

Fig. 4.

retina images from the same person taken under different circumstances may serve to examine the progress of diseases.

Finally, the quite important point related to the choice of the optimum parameter p of the distance d_p for enhancing pattern separability, can be done with a statistical approach using a learning sample of retina images and will be discussed in a forthcoming paper.

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